Finite Element Method Applied to Skin-Effect Problems in Strip Transmission Lines

GEORGE I. COSTACHE, SENIOR MEMBER, IEEE

Abstract —**This paper describes a two-dimensional finite element approach to the quasi-static TEM analysis of shielded or open conducting strips with applications to VLSI parasitic elements and transmission line characteristics of printed circuits. The approach uses a combination of two-dimensional and one-dimensional finite elements to solve the field problems in terms of the magnetic vector potential in the frequency domain. The method and the algorithm can be applied to shielded or open conducting strips and takes into account the skin effect and proximity effect between structures.**

The ac resistance and reactance calculated by using this approach can be used as input parameters to a circuit analysis program such as SPICE or similar programs.

I. Introduction

THE INCREASE in speed in modern integrated circuits and printed circuits demands better characterization of the electrical parameters which can influence the performance. At high and ultrahigh speed, when the tracks and connectors behave as transmission lines, the resistance increase due to the skin effect can be very important.

The finite element technique is one of the methods which can be applied to solve skin-effect problems in shielded or open conducting strip structures. A considerable amount of work has been done in solving two-dimensional skin-effect problems [1]–[3], reference [1] containing an excellent list of papers produced on these topics.

In this paper, the finite element method is applied to microstrip-like transmission line structures to calculate the ac resistance and reactance. The approach uses an integrodifferential finite element formulation [1], [2] in terms of the magnetic vector potential and assumes a TEM mode of propagation. It is based on a two-dimensional finite element solution of the linear diffusion equation, based on a Galerkin projection method. The algorithm presented in this paper can be applied to any arbitrary combination of conducting strips. The conducting strip width is divided in one-dimensional finite elements, while in the remaining region two-dimensional triangular finite element are used.

Manuscript received February 3, 1987; revised June 17, 1987. This work was supported by the Natural Science and Engineering Research Council of Canada.

The author is with the Ottawa-Carleton Institute for Electrical Engineering, Department of Electrical Engineering, Faculty of Engineering, University of Ottawa, Canada K1N 6N5.

IEEE Log Number 8716596.

The ac resistance and reactance derived by the finite element approach described in this paper take into account the current distribution inside the conducting strips due to the skin effect and proximity effect, and can be used for analysis of these structures in simulation models for multiple coupled transmission lines such as SPICE or other CAD tools [4].

II. Mathematical Formulation

The skin effect and proximity effect are taken into account by formulating the problem in terms of the magnetic vector potential for structures of microstrip-like transmission lines embedded in a dielectric substrate contained in a conducting box (see Fig. 1).

The following formulation is also valid for unbounded strip line structures by using the infinite element approach for the external boundary [5].

Let us consider the microstrip configuration of Fig. 1, where the conductors are carrying the alternating currents $i_k(t) = I_k\sqrt{2}\sin(\omega t + \alpha_k)$ $(k = 1, 2, \cdots, n)$ and are surrounded by an inhomogeneous dielectric.

The conductors are parallel to the z axis and have an electric conductivity σ and a cross section S_k $(k = 1, 2, \cdots, n)$. It is assumed that the magnetic permeability of the strips and of the surrounding medium is μ_0.

For the TEM mode of propagation, the complex magnetic vector potential has only one component in the z direction $A(x, y)$ which satisfies the partial differential equation

$$-\nabla^2 A(x, y) = \mu_0 J_k(x, y) \tag{1}$$

where $J_k(x, y)$ $(k = 1, 2, \cdots, n)$ is the current density inside each conducting strip, depending on the frequency and geometrical configuration. The electric field strength at any point is given by

$$\vec{E} = -j\omega\vec{A} - \nabla V \tag{2}$$

where $V(x, y, z)$ is the electric scalar component of the field. Inside the conducting strip, $E_x = E_y = 0$ and $A_x = A_y = 0$ and we can write

$$E(x, y) = -j\omega A(x, y) - dV/dz. \tag{3}$$

At the same time, at any point inside the conductor dV/dz

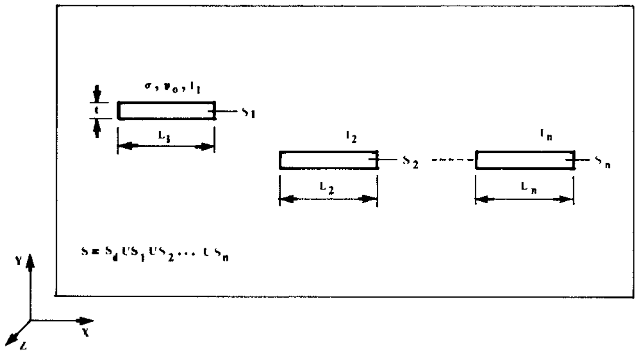


Fig. 1. Strip conductors in a box.

is a complex constant which depends on how the reference of the magnetic vector potential is chosen [6].

If an average value of the magnetic vector potential is defined over the cross section of each conducting strip by

$$\tilde{A} = \frac{1}{S_k} \int_{S_k} A(x, y)\, dx\, dy \tag{4}$$

then the current density for each point of any conducting strip is given by

$$J_k = -j\omega\sigma A + j\omega\sigma\tilde{A} + J_{0k} \tag{5}$$

where J_{0k} $(k = 1, 2, \cdots, n)$ is the average current density distribution defined as the ratio of the total current of the strip to its cross section S_k, $J_{0k} = I_k / S_k$ $(k = 1, 2, \cdots, n)$.

By replacing the current density in the initial equations (1), the general equations which take into account the skin effect and proximity effect of the conducting strips become

$$-\nabla^2 A + j\omega\mu_0\sigma A - j\omega\mu_0\sigma\tilde{A} = \mu_0 J_{0_k} \quad \text{on } S_k \tag{6}$$

$$-\nabla^2 A = 0 \quad \text{on } S_d \tag{7}$$

where S_k $(k = 1, 2, \cdots, n)$ is the cross section of each conducting strip and S_d is the region outside the strips.

To solve the above system of integrodifferential equations, the boundary conditions corresponding to the magnetic vector potential should be taken into account. For a shielded conducting strips structure, the value of the magnetic vector potential on the shield is constant if the shield is considered a perfect conductor. For simplicity, this value can be chosen equal to zero and the shield surface will correspond to a homogeneous Dirichlet boundary condition. When the shield is a magnetic material of high permeability, the corresponding boundary condition will be a homogeneous Neumann one. For open conducting strips, two approaches can be followed: the region can be bounded with an approximate Dirichlet or Neumann boundary or the infinite element approach can be used [5].

For use of the finite element method, (6) and (7) are put into a Galerkin integral form given by

$$\int_S (\nabla v)(\nabla A)\, dx\, dy + \sum_{k=1}^{n} \int_{S_k} \left[j\omega\mu_0\sigma(A - \tilde{A})v - \mu_0 J_{0k} v \right] dx\, dy = 0 \tag{8}$$

which must be satisfied for any continuous function $v(x, y)$, called the testing function. The solution of the integrodifferential equations can now be obtained by using a discretization of the Galerkin integrals by means of finite elements. This will lead us to construct a system of linear equations in terms of magnetic vector potential.

III. Finite Element Solution

The discretization of (8) can be carried out by using triangular finite elements. However, for conducting strip problems, the algorithm can be greatly simplified if the skin depth is greater than the thickness of the strip [8]. In this case, the current density will depend on the frequency only in the width direction.

In this situation, the skin effect in the direction of the conductor thickness is neglected. The above simplification, which is a good approximation for frequencies and dimensions of conducting strip structures used on printed circuit boards, changes the Galerkin integrals describing the field problem to the following expression:

$$\int_{S_d} (\nabla v)(\nabla A)\, dx\, dy + t \sum_{k=1}^{n} \int_{L_k} \left[(\nabla v)(\nabla A) + j\omega\mu_0\sigma(A - \tilde{A})v - \mu_0 J_{0k} v \right] dx = 0. \tag{9}$$

To simplify the notation, the following description refers to one shielded conducting strip only, as shown in Fig. 2.

In (9), $(\nabla A)^2$ is the integrand of two integrals: the first extends over the dielectric region, while the second corresponds to the conducting strip cross section where the magnetic vector potential and consequently the current density depend only on the width direction. Following this approach, a combination of triangular and linear finite elements is used. The dielectric region is divided into two-dimensional finite elements while one-dimensional finite elements are used for the conducting strips.

The trial function used for approximating the magnetic vector potential on each triangle is a linear combination of polynomials:

$$A(x, y) = \sum_{i=1}^{3} \alpha_i(x, y) A_i \tag{10}$$

where $\alpha_i(x, y)$ are linear forms and A_i are the node potentials [1].

The trial function used to approximate the magnetic vector potential on the conducting strip is given by

$$A(s) = \beta_i(s) A_i + \beta_j(s) A_j = \frac{s_j - s}{s_j - s} A_i + \frac{s - s_i}{s_j - s_i} A_j \tag{11}$$

where s is a variable along the strip width, s_i and s_j denote the extension of the linear finite element, and A_i, A_j are the node values corresponding to this element. The linear functions multiplying the node potentials in (10) and (11) are called shape functions.

The integral form (9) can now be written as a summation over all triangles and linear elements. By using as testing functions $v(x, y)$, the shape functions defined in (10) and (11) for triangular elements and linear elements,

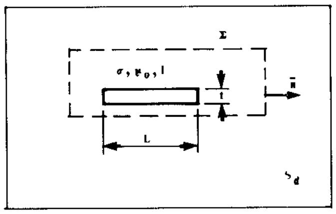


Fig. 2. Shielded conducting strip.

respectively, a final system of equations is obtained with respect to the magnetic vector potential of each node, its general form being given by

$$[S_1 + j\omega\mu_0\sigma S_2]A = b. \tag{12}$$

To evaluate the average value of the magnetic vector potential needed for computation, the connectivity coefficients are introduced for the one-dimensional finite elements. These coefficients are defined as follows:

$$c_{ik} = \begin{cases} 1 & \text{if the } i \text{ node belongs to the } k \text{ segments} \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Using these coefficients, the average magnetic vector potential for one conducting strip can be written as

$$\tilde{A} = \frac{1}{2L} \sum_{k=1}^{NOS} \sum_{i=1}^{NP} l_k c_{ik} A_i \tag{14}$$

where NP is the total number of points on the strip, NOS is the number of linear finite elements in which the conducting strip is divided, L is the width of the strip, and l_k is the length of each linear finite element.

IV. Conducting Strip Resistance and Reactance

Once the magnetic vector potential is calculated, one can evaluate the current distribution by using [3]:

$$J = -j\omega\sigma A + j\omega\sigma\tilde{A} + J_0. \tag{15}$$

The ratio between ac resistance and dc resistance can be written as

$$k_r = \frac{R_{ac}}{R_{dc}} = \frac{P}{I^2 L}\sigma S \tag{16}$$

where $P = RI^2$ is the power dissipated in the conductor, L is its length, and S its cross section. The power per unit length may be calculated as a real part of the flux of the Poynting vector through a cylindrical surface Σ (see Fig. 2) which contains the unit length of the conducting strip. Starting with the relation

$$\frac{P}{L} = \mathrm{Re}\left\{ \int_{\Sigma} (\bar{E} \times \bar{H}^*)\bar{n}\, ds \right\} \tag{17}$$

one can show that the ratio between ac resistance and dc resistance is given by [6]:

$$k_r = 1 + \mathrm{Re}\left\{ j\frac{\omega\sigma t L}{I}\tilde{A} \right\}. \tag{18}$$

In the same way, the ratio between reactance and dc resistance can be obtained and is given by the following relation:

$$k_x = \mathrm{Im}\left\{ j\frac{\omega\sigma t L}{I}\tilde{A} \right\}. \tag{19}$$

From the above formulation, starting from the magnetic vector potential, one can calculate the current distribution along the conducting strip, the average value of the magnetic vector potential, and, finally, the ratio between ac and dc resistance and the ratio between reactance and dc resistance. Since the finite element formulation is not restricted by the geometry, the approach can be used for any arbitrary conducting strip configuration. Other parameters of interest, such as capacitance per unit length, can be obtained by using a finite element approach as described in [9].

The approach described above has been implemented in a Fortran program and has been applied to a few examples.

V. Examples and Numerical Results

Two configurations of shielded microstrips have been solved. The first example was used to test the program; it consists of a symmetrical shielded microstrip problem previously solved by an analytical technique based on a finite Fourier transform [7]. Fig. 3 shows the configuration of the first example.

Since the purpose of this calculation was to compare the results obtained by the finite element method with those published in [7] and calculated with the Fourier transform, the calculations were performed for the same example as described in [7]. The finite element method using two-dimensional and one-dimensional elements was applied to a shielded strip made of copper with the following parameters: $g = 5$ mm, $b = 55$ mm, $h = 65$ mm, and $l = 12.5$ mm. The calculations were done for a frequency of 50 Hz.

One of the coarse triangularizations used for calculations is shown in Fig. 4.

For this specific division into finite elements, the current density distribution was obtained and compared with the published results obtained by the finite Fourier transform. Table I shows a comparison between the normalized current density obtained by taking the ratio between current density and dc current distribution for the nodes 1,2,3,4,5 located on the conducting strip (Fig. 4).

It should be noticed that by using only 30 triangles and five free nodes on the linear finite elements located on the strip, the results obtained by the finite element method are relatively close to those obtained by a Fourier transform method. With a more refined division into triangles and linear finite elements (20 free nodes), the results obtained by the finite element approach were within 1 percent when compared to those produced by the Fourier transform method.

The second example solved by using the program based on the algorithm described in this paper is the multiple-strip configuration shown in Fig. 5.

For simplicity, the current was assumed in opposite directions for two adjacent lines and the structure is con-

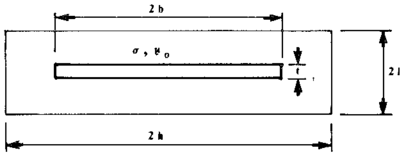


Fig. 3. Cross section of a symmetrical shielded strip

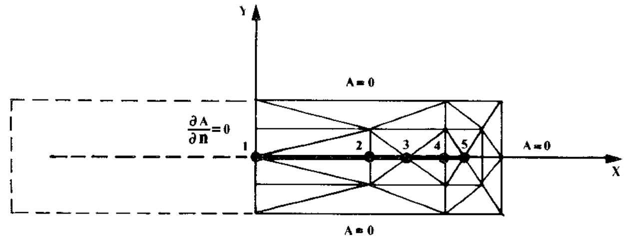


Fig. 4. Division into finite elements.

TABLE I
COMPARISON BETWEEN NORMALIZED CURRENT DENSITY CALCULATED BY FINITE ELEMENTS AND FINITE FOURIER TRANSFORM

Node	Distance x[mm]	Finite element solution	Fourier transform [7]
1	0	0.9732 − j 0.0554	0.9805 − j 0.0374
2	30	1.0046 − j 0.0342	0.9897 − j 0.0307
3	40	1.0051 − j 0.0061	1.0089 − j 0.0059
4	50	1.0204 + j 0.1102	1.0490 + j 0.1133
5	55	1.0421 + j 0.2515	1.0510 + j 0.2811

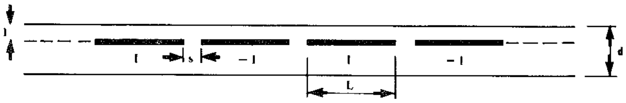


Fig. 5 Multiple-strip configuration.

sidered periodic. The geometrical parameters considered for this example are $t = 5$ μm, $L = 1100$ μm, $s = 200$ μm, $l = 125$ μm, and $d = 500$ μm. Due to the periodic structure, only one cell corresponding to one conducting strip was solved by replacing the separating wall between two of them with a homogeneous Dirichlet boundary condition. Table II shows the current density distribution along one strip for different frequencies and distances measured from the center of the strip.

The ratio between ac resistance and dc resistance and the ratio between reactance and dc resistance, which are also incorporated in the program, have been calculated and some of the results are presented in Table III.

The above algorithm and program can be applied successfully only in situations where the thickness of the strip is small compared to the equivalent penetration depth. For situations where the equivalent penetration depth is comparable to the thickness of the strip, the algorithm should change in order to accommodate two-dimensional finite elements inside the conducting strip.

TABLE II
NORMALIZED CURRENT DISTRIBUTION FOR DIFFERENT POINTS ON THE STRIP AT VARIOUS FREQUENCIES

DISTANCE	NORMALIZED CURRENT DENSITY		
μm	f = 1MHz	f = 5 MHz	f = 10 MHz
0	0.996	0.948	0.877
200	0.997	0.956	0.890
300	0.998	0.968	0.910
400	1.001	1.032	1.062
500	1.006	1.124	1.281
550	1.011	1.231	1.557

TABLE III
RESISTANCE AND REACTANCE RATIO

f[MHz]	$k_r = \frac{R_{a.c.}}{R_{d.c.}}$	$k_r = \frac{X}{R_{d.c.}}$
1	1.011	0.171
5	1.262	0.767
10	1.788	1.169
15	2.267	1.269

VI. CONCLUSIONS

The integrodifferential finite element method has been applied to the skin-effect problem in shielded conducting strips by using the combination of one- and two-dimensional elements. The results obtained by this approach are in agreement with those obtained by other methods, such as the finite Fourier transform.

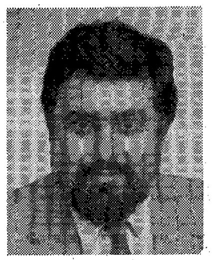
ACKNOWLEDGMENT

The author wishes to thank the anonymous reviewer who made very useful remarks about the paper.

REFERENCES

[1] A. K. Konrad, "Integrodifferential finite element formulation of two-dimensional steady-state skin-effect problems," *IEEE Trans. Magn.* vol. MAG-18, no. 1, pp. 284–292, 1982.
[2] G. Costache, "Finite element solution of steady-state skin-effect problems in straight flat conductors," *COMPEL—Int. J. Comput. and Math. in Elec. and Electron. Eng.*, vol. 2, no. 2, pp. 35–39, 1983.
[3] G. Costache, "Calculation of eddy currents and skin-effect in nonmagnetic conductors by finite element method," *Rev. Roum. Sci Tech., Ser. Electrotech. Energ.*, vol. 21, no. 3, pp. 357–363, 1976.
[4] V. J. Tripathi and J. B. Rettig, "A SPICE model for multiple coupled microstrips and other transmission lines," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-33, pp. 1513–1518, Dec. 1985.
[5] S. Pissanetzky, "A simple infinite element," *COMPEL—Int. J. Comput. and Math. in Elec. and Electron. Eng.*, vol. 3, no. 2, pp. 107–114, 1984.
[6] A. Timotin, "Iteration method for study of the skin-effect in straight conductors," *Rev Roum. Sci. Tech., Ser. Electrotech. Energ.*, vol. 10, no 1, pp. 19–45, 1965

[7] A. Timotin and A. Nicolae, "Skin-effect in shielded straight flat conductors," *Rev. Roum. Sci. Tech., Ser. Electrotech. Energ.*, vol. 17, no. 1, pp. 3–22, 1972.
[8] P. Waldow and I. Wolf, "The skin-effect at high frequencies," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-33, pp. 1076–1082, Oct. 1985.
[9] Z. Pantic and R. Mittra, "Quasi-TEM analysis of microwave transmission lines by the finite-element method," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-34, pp. 1096–1103, Nov. 1986.



George I. Costache (M'78–SM'82) is an Associate Professor of Electrical Engineering at the University of Ottawa, Ottawa, Ont., Canada. His career has included positions with Bell-Northern Research in Ottawa, the University of Manitoba, Electricite de France, and the Polytechnical Institute of Bucharest. He has taught electromagnetics and numerical techniques applied to electromagnetics for more than 19 years and has made original contributions to the solution of skin-effect problems and transient phenomena. His main interest is in numerical techniques, such as finite-element analysis and moment methods, and in their applications to electromagnetic interference and compatibility problems.

Dr. Costache is the author or coauthor of over 50 technical papers and reports. He is a member of the editorial review board of *COMPEL, The International Journal for Computation and Mathematics in Electrical and Electronics Engineering*. He is a registered Professional Engineer in the province of Ontario, Canada.